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Effective Estimates of Invertibility Intervals for Linear Multipoint Boundary Value Problems*

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Inequalities and explicit formulas are given for Green's function for multipoint boundary value problems. These relations are combined with a new method for studying uniqueness problems to obtain explicit criteria for uniqueness in terms of the operator coefficients.

1. INTRODUCTION

Developed in this paper are some new methods for the study of the uniqueness problem for an nth order linear ordinary differential equation with de la Vallee Poussin boundary conditions:

$$Ly = 0, \quad y^{(i)}(s_j) = 0, \qquad 0 \leqslant i \leqslant r_j - 1, \quad 0 \leqslant j \leqslant \mu. \tag{1.1}$$

It is assumed that L is regular and has continuous coefficients on $-\infty < t < \infty$, further, $s_0 < \cdots < s_\mu$ and $r_0 + \cdots + r_\mu = n$.

Communicated in this article is a new method called *partial inversion*, which is used to obtain *effective criteria* for uniqueness, i.e., inequalities involving the interval $[a, b]$ and the coefficients of L. The techniques involve inversion of boundary value problems via Green's function, some calculus inequalities and detailed inequalities and identities for the spectral radius of the linear integral operator associated with Green's function.

The central ideas of the paper are best illustrated by the example

$$Ly = y^{(6)} - \sum_{i=0}^{2} p_i(t)\, y^{(i)} = 0,$$

$$y(a) = y'(a) = y''(a) = y'''(a) = y(b) = y'(b) = 0,$$

with $p_i \in C[a, b]$, $0 \leqslant i \leqslant 2$. Assuming nonuniqueness for this problem, we decompose L as $L = KM - Q$ where

$$K = (d/dt)^4, \qquad M = (d/dt)^2, \qquad Q = \sum_{i=0}^{2} p_i(t)(d/dt)^i.$$

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The operators K and M are disconjugate on $[a, b]$, and we find that $u(t) = [My](t)$ satisfies

$$Ku = f(t),$$
$$u(a) = u'(a) = u(b_1) = u(b_2) = 0,$$

where $a < b_1 < b_2 < b$. Inverting in terms of Green's function $\mathscr{G}$ for the preceding problem we have

$$My = \int_a^{b_2} \mathscr{G}(t, s)[Qy](s)\, ds. \tag{1.2}$$

Using some basic estimates (Sections 6, 7) we arrive at the necessary condition

$$1 \leqslant \frac{377.4}{10^6}(b-a)^6 \| p_0 \| + \frac{154.91}{10^5}(b-a)^5 \| p_1 \| + \frac{9}{2048}(b-a)^4 \| p_2 \|. \tag{1.3}$$

It is the process of arriving at (1.2) that we call *partial inversion.*The procedure is useful only under certain kinds of hypotheses (see Sections 3, 4, 5). In going from (1.2) to (1.3) it was necessary to estimate $| Qy |$ in terms of $\max\{| My |: a \leqslant t \leqslant b_2\}$ and also to estimate the spectral radii of various linear integral operators generated from $\mathscr{G}$. The objective of such estimates is to free the final necessary condition (1.3) from dependence on $\mathscr{G}$, b, b_2, y.

The motivation for the study of uniqueness of problem (1.1) comes from many sources. One such source is the conversion of nonlinear boundary value problems to integral equations of Fredholm type. The impact of uniqueness of (1.1) on this problem is seen through the existence of a Green's function integral kernel for the nonhomogeneous problem associated with (1.1). Another motivating problem is the convergence theory for numerical methods associated with nonlinear equations of the form $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$. Such methods often depend for convergence upon the uniqueness of a linear variational problem of the form (1.1). Finally, we mention the classical extrema theory in the calculus of variations, for which conjugate point hypotheses are introduced; often such hypotheses reduce to uniqueness questions for (1.1).

2. Notation, Definitions

A linear differential operator $K: C^k[a, b] \to C[a, b]$,

$$Ku = u^{(k)} + \sum_{i=0}^{k-1} p_i(t)\, u^{(i)},$$

is *disconjugate* on $[a, b]$ iff the only solution of $Ku = 0$ with k zeros on $[a, b]$ counting multiplicities is $y \equiv 0$. Polya [17] has shown that this notion is equivalent to the factorization of K into first order operators; see also Hartman [10] and Coppel [5].

Given $x \in \mathbb{R}^k$, $x = (x_1, \ldots, x_k)$, the *boundary operator* $\mathscr{L}_x$ is defined as follows:

$$\mathscr{L}_x y = (y^{(m_1)}(x_1), \ldots, y^{(m_k)}(x_k))^T.$$

In this relation, $m_i + 1 =$ the number of indices $j \leqslant i$ with $x_j = x_i$. The set comp(x) consists of all the distinct components $t_0, \ldots, t_\nu$ of x; usually $t_0 < \cdots < t_\nu$. The smallest interval $[a, b]$ containing $t_0, \ldots, t_\nu$ is the convex hull co{comp(x)}, therefore

$$[a, b] = [t_0, t_\nu] = \mathrm{co}\{\mathrm{comp}(x)\}.$$

Unless otherwise stated, all three notations are simultaneously in use.

The condition $\mathscr{L}_x y = 0$ in the above notation is the set of *de la Vallee Poussin conditions*

$$y^{(i)}(t_j) = 0, \qquad 0 \leqslant i \leqslant n_j - 1, \quad 0 \leqslant j \leqslant \nu,$$

where n_j is the number of repetitions of t_j in the vector x. *Hereafter*, $n_x(S)$ denotes the number of components of x in the set S; in particular, $n_j = n_x(t_j)$, $0 \leqslant j \leqslant \nu$, and $n_x[a, b] = k$. The symbol $n_x(S)$ makes it convenient to regard singleton sets as points and to use interval notation $[c, d]$ even in case $c = d$.

Considered in this paper are three differential operators

$$Ku = u^{(k)} + \sum_{i=0}^{k-1} p_i(t)\, u^{(i)};$$

$$My = y^{(m)} + \sum_{j=0}^{m-1} a_j(t)\, y^{(j)};$$

$$Qy = \sum_{j=0}^{n-1} q_j(t)\, y^{(j)}.$$

In these relations, $m + k = n$, $p_i \in C(\mathbb{R})$, $a_j \in C^k(\mathbb{R})$, $q_r \in C(\mathbb{R})$, $0 \leqslant i \leqslant k - 1$, $0 \leqslant j \leqslant m - 1$, $0 \leqslant r \leqslant n - 1$. The coefficients of Q may all vanish. The product KM is defined on $C^n(\mathbb{R})$.

The operator L of (1.1) will be assumed to have the form $L = KM - Q$, with M disconjugate on $[a, b]$. Later, it will be assumed that K is diconjugate on $[a, b]$ as well. There are, of course, many ways of writing down such a decomposition of L. It is assumed that K is known explicitly, along with its kernel, while only the coefficients of M are known.

The Green's function for $Ku = f$, $\mathscr{L}_x u = 0$ will be denoted by $G(t, s; x)$. When $K = (d/dt)^k$, we replace G by $\mathscr{G}$. A more complete discussion of G appears in Section 6.

It has proved convenient to use $\wedge$ and $\vee$ for "min" and "max" in various places. For notation and definitions not given explicitly, see Hartman [10].

3. Inherited Boundary Operators

Let $M: C^{m+k}[a, b] \to C^k[a, b]$ be disconjugate of order m, $x \in [a, b]^{m+k}$. The zero properties of the unique solution y^* of the problem $(d/dt)^k My^* = 1$, $\mathscr{L}_x y^* = 0$ are well-known. In particular, My^* has exactly k zeros in $[a, b]$, and those of higher multiplicity lie in comp(x). Using this situation as a model, the following real variable lemma isolates similar facts, which are to be used in connection with operator products KM, K not necessarily disconjugate.

Lemma 3.1. *Let* $M: C^{m+k}[a, b] \to C^k[a, b]$ *be disconjugate. Suppose* $x \in [a, b]^{m+k}$, $y \in C^{m+k}[a, b]$ *and* $\mathscr{L}_x(y) = 0$. *Then there exists* $v \in [a, b]^k$ *with* $\mathscr{L}_v(My) = 0$ *having the following properties*:

(a) *If* $s \in \operatorname{comp}(x)$ *and* $n_x(s) \geqslant m + 1$, *then* $s \in \operatorname{comp}(v)$ *and* $n_v(s) + m = n_x(s)$.

(b) *If* $c, d \in \operatorname{comp}(x)$, $c < d$ *and* $n_x[c, d] \geqslant m$, *then*

$$n_v[c, d] + m = n_x[c, d].$$

(c) *If* $s \in \operatorname{comp}(v)$ *and* $n_v(s) > 1$, *then* $s \in \operatorname{comp}(x)$ *and* $n_v(s) + m = n_x(s)$.

(d) $\operatorname{comp}(v) \subseteq \operatorname{co}\{\operatorname{comp}(x)\}$.

Proof. Suppose $\operatorname{comp}(x) = \{s_0 < \cdots < s_\nu\} \subseteq [a, b]$ and let σ_i be the number of s_j with $n_x(s_j) = i$, $1 \leqslant i \leqslant m + k$. Define $\tau_i = \sigma_i + \cdots + \sigma_{m+k}$; then $\sum_{i=1}^{m+k} \tau_i = \sum_{i=1}^{m+k} r\sigma_r = n_x[a, b] = m + k$. Via Rolle's theorem, the generalized derivatives $y^{[l]}$ associated with M can be shown by induction to have the following zero properties: $y^{[l]}$ has $R_l \equiv \sum_{i=1}^{l+1} \tau_i - l$ distinct zeros in $[a, b]$, $0 \leqslant l \leqslant m$. In particular, $y^{[m]} = My$ has R_m distinct zeros in $[a, b]$, among which we include those s_j satisfying $n_x(s_j) \geqslant m + 1$. The total multiplicity $\mathscr{M}$ assigned at these R_m distinct points by this process is

$$\mathscr{M} = R_m + \sum_{j=0}^{\nu} \max\{n_x(s_j) - m - 1, 0\}$$

$$= R_m + \sum_{r=m+2}^{m+k} (r - m - 1)\sigma_r = \sum_{r=1}^{m+k} r\sigma_r - m = n_x[a, b] - m = k.$$

Let $v \in [a, b]^k$ be any vector whose components are the R_m distinct points constructed above, with $n_v(s_j) = n_x(s_j) - m$ whenever $n_x(s_j) \geqslant m + 1$. Then (a), (c) and (d) hold automatically. To prove (b), observe that the special case $c = a$, $d = b$ has been proved above. Replace k by $n_x[c, d]$, and x by the vector $\bar{x}$ obtained from x by deleting those components outside $[c, d]$; then the special case already proved implies (b). ∎

DEFINITION. The operator $\mathscr{L}_v$ described in Lemma 3.1 is called *the boundary operator inherited from* $\mathscr{L}_x$, M *and* y. The *inherited interval* I_y is the convex hull of comp(v).

4. m-HEREDITARY OPERATORS

The purpose of this section is to isolate a necessary and sufficient condition on $x \in [a, b]^{m+k}$ such that $n_x(I_y) \geqslant m$. In order to be useful, the condition has to be independent of the particular mth order disconjugate operator M and also independent of $y \in C^{m+k}[a, b]$.

LEMMA 4.1. *Let* $x \in [a, b]^{m+k}$. *A given* mth *order disconjugate operator* M: $C^{m+k}[a, b] \to C^k[a, b]$ *satisfies* $n_x(I_y) \geqslant m$ *for every* $y \in C^{m+k}[a, b]$, $\mathscr{L}_x y = 0$, *iff there exists an interval* $[c, d] \subseteq [a, b]$ ($c = d$ *is allowed*) *such that* $n_x[a, c] \geqslant m + 1$, $n_x[c, d] \geqslant m$, $n_x[d, b] \geqslant m + 1$.

Proof. Suppose $[c, d] \subseteq [a, b]$, $n_x[a, c] \geqslant m + 1$, $n_x[c, d] \geqslant m$, $n_x[d, b] \geqslant m + 1$. Given any $y \in C^{m+k}[a, b]$ with $\mathscr{L}_x y = 0$ it follows from Lemma 3.1 that My has a zero in $[a, c]$ and a zero in $[d, b]$. Therefore, $[c, d] \subseteq I_y$ and $n_x(I_y) \geqslant n_x[c, d] \geqslant m$.

Conversely, let y have m zeros in I_y for all $y \in C^{m+k}[a, b]$. Define $c = \inf\{t \geqslant a: n_x[a, t] \geqslant m + 1\}$, $d = \sup\{s \leqslant b: n_x[s, b] \geqslant m + 1\}$. It follows that $c, d \in$ comp(x).

Assume $c > d$. Then $[a, c]$ and $[d, b]$ are nondegenerate closed intervals on which M is disconjugate. Select $\alpha, \beta \in \mathbb{R}^m$ as follows: α is obtained from x by deleting the components of x in $[c, b]$ and adjoining c exactly $m - n_x[a, c)$ times; β is obtained from x by deleting the components of x in $[a, d]$ and adjoining d exactly $m - n_x(d, b]$ times. Let y_0 and y_1 satisfy $My_0 = My_1 = 1$, $\mathscr{L}_\alpha y_0 = \mathscr{L}_\beta y_1 = 0$. Denote by y any function in $C^{m+k}[a, b]$ such that $y = y_0$ on $[a, c - \epsilon]$, $y = y_1$ on $[c, b]$, where $\epsilon > 0$ is so small that $[c - \epsilon, c] \cap$ comp$(x) = \{c\}$. Since $My \neq 0$ on $[a, c - \epsilon] \cup [c, b]$ and $\mathscr{L}_x y = 0$, $I_y \subseteq (c - \epsilon, c)$, therefore $m \leqslant n_x(I_y) \leqslant n_x(c - \epsilon, c) = 0$, a contradiction. This proves $c \leqslant d$.

To complete the proof, suppose first that $a < c \leqslant d < b$. Construct y_0 and y_1 as in the preceding paragraph. Define y to be y_0 on $[a, c - \epsilon]$ and y_1 on $[d + \epsilon, b]$, where $\epsilon > 0$ is sufficiently small. Let y be defined on $[c - \epsilon, d + \epsilon]$ as a C^∞-connection of y_0 and y_1 (or $-y_0$ and y_1) in such a way that $\mathscr{L}_x y = 0$ and all zeros of y lie in comp(x). Since $My \neq 0$ on $[a, c - \epsilon] \cup [d + \epsilon, b]$, it follows that $I_y \subseteq (c - \epsilon, d + \epsilon)$. Therefore, $m \leqslant n_x(I_y) \leqslant n_x[c - \epsilon, d + \epsilon]$ for every small $\epsilon > 0$. This proves that $n_x[c, d] \geqslant m$.

The remaining cases are trivial, because $n_x[c, d] \geqslant \max\{n_x(c), n_x(d)\} \geqslant m + 1$ when either $c = a$ or $d = b$. ∎

DEFINITION 4.2. A point $x \in [a, b]^{m+k}$, or the corresponding boundary operator $\mathscr{L}_x$, is called *m-hereditary* iff there exists an interval $[c, d] \subseteq [a, b]$ ($c = d$ is allowed) such that $n_x[a, c] \geqslant m + 1$, $n_x[c, d] \geqslant m$, $n_x[d, b] \geqslant m + 1$.

5. PARTIAL INVERSION

Consider the three ordinary differential operators K, M, Q of order k, m, q, respectively, described earlier in the paper. Given $x \in \mathbb{R}^{m+k}$, $[a, b] = \text{co}\{\text{comp}(x)\}$, the purpose of this section is to solve for My in the problem

$$KMy - Qy = 0, \qquad \mathscr{L}_x y = 0, \tag{5.1}$$

via appropriate use of Green's function kernels.

Partial Inversion Lemma

LEMMA 5.1. *Let M be disconjugate on $[a, b]$. Assume $\mathscr{L}_x$ is m-hereditary and y is a solution of* (5.1) *on $[a, b]$.*

If $\mathscr{L}_v$ is the boundary operator inherited from $\mathscr{L}_x$, M and y with inherited interval $[c, d]$, and $Ku = 0$, $\mathscr{L}_v u = 0$ has only the solution $u \equiv 0$, then

$$[My](t) = \int_0^d G(t, s; v)[Qy](s)\, ds, \qquad c \leqslant t \leqslant d, \tag{5.2}$$

where $G(t, s; v)$ is the Green's function for the problem $Ku = f(t)$, $\mathscr{L}_v u = 0$.

Proof. Put $u(t) = [My](t)$, then $Ku = Qy$, $\mathscr{L}_v u = 0$, hence the cited inversion formula. ∎

Some remarks concerning (5.2) are in order. First, this relation is valid without regard to whether or not $\mathscr{L}_x$ is m-hereditary. It turns out, however, that (5.2) is of no use unless y has m zeros in $[c, d]$. Secondly, K need not be disconjugate to write down (5.2). However, in most cases of any practical interest K will be disconjugate. Finally, the location of comp(v) is only partially known: comp(v) $\subseteq$ co{comp(x)} and repeated entries of v belong to comp(x). Any useful criterion arising from (5.2) must in its development be freed from dependence on v.

6. GREEN'S FUNCTION

Let K: $C^k[a, b] \to C[a, b]$ be a kth order linear ordinary differential operator. If $x \in \mathbb{R}^k$, co{comp(x)} $= [a, b]$ and the problem $Ku = 0$, $\mathscr{L}_x u = 0$ has only the zero solution $u \equiv 0$, then the problem $Ku = f$, $\mathscr{L}_x u = 0$ has the unique solution $u(t) = \int_a^b G(t, s; x) f(s)\, ds$ for each $f \in C[a, b]$. The kernel $G(t, s; x)$ is called the *Green's function* for K and $\mathscr{L}_x$ on $[a, b]$. Its fundamental properties may be found in Coppel [5].

Two particularly useful representations of G will be described below, for later exploitation within this paper. The first duplicates exactly ordinary computation of G; the second is more useful for establishing properties of G.

Let $u_1, \ldots, u_k$ be a fixed basis for $\ker(K)$, and set $U(t) = (u_1(t), \ldots, u_k(t))$, $W(t) = [u_j^{(i-1)}(t)]$, $1 \leqslant i, j \leqslant k$, $e = (0, \ldots, 0, 1)^T \in \mathbb{R}^k$. Assume $\operatorname{comp}(x) = \{t_0 < \cdots < t_\nu\}$ and put $x^* = (t_0, \ldots, t_0, t_1, \ldots, t_1, \ldots, t_\nu, \ldots, t_\nu)$, with $n_{x^*}(t_i) = n_x(t_i)$, $0 \leqslant i \leqslant \nu$. Define $E_i = [t_0, t_i]$, $0 \leqslant i \leqslant \nu$, and put

$$V_x(s) = \operatorname{diag}\{\chi_{E_0} I_{n_x(t_0)}, \ldots, \chi_{E_\nu} I_{n_x(t_\nu)}\},$$

where I_n is the $n \times n$ identity matrix, $\chi_E(t) = 1$ for $t \in E$, $\chi_E(t) = 0$ for $t \notin E$. Set $\epsilon(u) = 1$ for $u \geqslant 0$, $\epsilon(u) = 0$ for $u < 0$, and put $k(t, s) = U(t) W^{-1}(s) e$, $\mathscr{K}(t, s) = \epsilon(t - s)\, k(t, s)$.

The two formulas for G are as follows:

$$G(t, s; x) = U(t)[\epsilon(t - s) I - \mathscr{L}_{x^*}(U)^{-1} V_x(s)\, \mathscr{L}_{x^*}(U)]\, W^{-1}(s) e,$$
$$G(t, s; x) = \mathscr{K}(t, s) - U(t)\, \mathscr{L}_{x^*}(U)^{-1} \mathscr{L}_{x^*}[\mathscr{K}(\cdot, s)].$$

In either formula, $a = t_0 \leqslant t, s \leqslant t_\nu = b$. The connection between the two formulas is given by the identity $\mathscr{L}_{x^*}[\mathscr{K}(\cdot, s)] = V_x(s)\, \mathscr{L}_{x^*}(U)\, W^{-1}(s) e$.

LEMMA 6.1. *Let $x \in \mathbb{R}^k$, $\operatorname{co}\{\operatorname{comp}(x)\} = [x, b]$ and assume $G(t, s; x)$ exists. Then for all $y \in \mathbb{R}^k$, $\operatorname{co}\{\operatorname{comp}(y)\} = [a, b]$, and $\| y - x \|$ sufficiently small, $G(t, s; y)$ exists and*

$$\lim_{y \to x} \frac{\partial^i}{\partial t^i} [G(t, s, x) - G(t, s; y)] = 0, \qquad 0 \leqslant i \leqslant k,$$

uniformly on $[a, b] \times [a, b]$. (See Gustafson [8]).

LEMMA 6.2. *Let the formal adjoint K^* of the operator K be defined. Assume K disconjugate on $[a, b]$ and denote by v, v^*, w, w^* solutions of $Kv = Kw = K^*v^* = K^*w^* = 0$ such that:*

(a) *v and v^* have zeros of order $(l, k - l - 1)$ and $(k - l - 1, l)$ at $\{a, b\}$, resp., with $(-1)^{k-l-1} v^{(k-l-1)}(b) = v^{*(k-l-1)}(a) = 1$.*

(b) *w and w^* have zeros of order $(l - 1, k - l)$ and $(k - l, l - 1)$ at $\{a, b\}$, resp., with $(-1)^{l-1} w^{*(l-1)}(b) = w^{(l-1)}(a) = 1$.*

Then the Green's function $G(t, s)$ for the problem $Ku = f$, $u^{(i)}(a) = u^{(j)}(b) = 0$, $0 \leqslant i \leqslant l - 1$, $0 \leqslant j \leqslant k - l - 1$, satisfies the inequality

$$| G(t, s)| \leqslant \min \left\{ \frac{v(t)\, v^*(s)}{v^{(l)}(a)}, \frac{w(t)\, w^*(s)}{| w^{(k-l)}(b)|} \right\}, \ (t, s) \in [a, b] \times [a, b].$$

(See Bates and Gustafson [1].)

Let $e_1, \dots, e_k$ be the standard unit vectors in $\mathbb{R}^k$. Given $x \in \mathbb{R}^k$, $[a, b] = \mathrm{co}\{\mathrm{comp}(x)\}$, $a = t_0 < \cdots < t_\nu = b$ the distinct components of x, define $x^* = (t_0, \dots, t_0, \dots, t_\nu, \dots, t_\nu)$, where $n_{x^*}(t_i) = n_x(t_i)$, $0 \leqslant i \leqslant \nu$. Put $n_0 = n_x(t_0), \dots, n_\nu = n_x(t_\nu)$.

LEMMA 6.3. *With the notation above, let K be a disconjugate operator on $[a, b]$ and denote by $v_i(t)$ the solution of the problem $Kv = 0$, $\mathscr{L}_{x^*}v = e_{n_0+\cdots+n_i}$, $0 \leqslant i \leqslant \nu$. The Green's function $G(t, s; x)$ satisfies the following estimates*:

$$\text{(a)} \quad |G(t, s; x)| \leqslant \min \left\{ |v_0(t)| \cdot \frac{|G^{(n_\nu)}(t_\nu, s; x)|}{|v_0^{(n_\nu)}(t_\nu)|}, |v_\nu(t)| \frac{|G^{(n_0)}(t_0, s; x)|}{|v_\nu^{(n_0)}(t_0)|} \right\};$$

$$\text{(b)} \quad |G(t, s; x)| \leqslant \max \left\{ \frac{|G^{(n_0)}(t_0, s; x)|}{|v_i^{(n_0)}(t_0)|}, \frac{|G^{(n_\nu)}(t_\nu, s; x)|}{|v_i^{(n_\nu)}(t_\nu)|} \right\} \cdot |v_i(t)|, \qquad 0 < i < \nu.$$

Proof. Let $v = v_i$, $0 \leqslant i \leqslant \nu$ fixed. Fix $s \in [a, b] \backslash \mathrm{comp}(x)$, and define $y(t) = G(t, s; x)/v(t)$ for $t \in [a, b] \backslash \mathrm{comp}(x)$, $y(t_j) = G^{(n_j)}(t_j, s; x)/v^{(n_j)}(t_j)$ for $j \neq i$, $y(t_i) = 0$. By the differential properties of G, the disconjugacy of K and L'Hospital's Rule, $y(t)$ is continuous and well-defined on $a \leqslant t \leqslant b$. The degenerate cases $n_i = 1$, $i = 0$, τ, which follow similarly, will be excluded hereafter.

The function $y(t)$ is continuously differentiable on $[a, b] \backslash \mathrm{comp}(x)$ due to the differential properties of G. If $\nu > 1$ and $t_j \in (a, b)$, then $n_j \leqslant n - 2$, so $G^{(n_j+1)}(t, s; x)$ is continuous for $|t - t_j| < \delta_j$, $\delta_j > 0$. This implies that

$$G(t, s; x) = \frac{(t - t_j)^{n_j}}{n_j!} \Big[G^{(n_j)}(t_j, s; x) + (t - t_j) \int_0^1 G^{(n_j+1)}(\lambda t + (1 - \lambda) t_j, s; x)(1 - \lambda)^{n_j} \, d\lambda \Big]$$

for $|t - t_j| < \delta_j$. A similar representation holds for $v(t)$. Therefore,

$$y(t) = \frac{G^{(n_j)}(t_j, s; x) + (t - t_j) \int_0^1 G^{(n_j+1)}(\lambda t + (1 - \lambda) t_j, s; x)(1 - \lambda)^{n_j} \, d\lambda}{v^{(n_j)}(t_j) + (t - t_j) \int_0^1 v^{(n_j+1)}(\lambda t + (1 - \lambda) t_j)(1 - \lambda)^{n_j} \, d\lambda}$$

for $j \neq i$, while if $t_i \in (a, b)$, then

$$y(t) = \frac{(t - t_i)}{n_i} \cdot \frac{G^{(n_i)}(t_i, s; x) + (t - t_i) \int_0^1 G^{(n_i+1)}(\lambda t + (1 - \lambda) t_i, s; x)(1 - \lambda)^{n_i} \, d\lambda}{1 + (t - t_i) \int_0^1 v^{(n_i)}(\lambda t + (1 - \lambda) t_i)(1 - \lambda)^{n_i - 1} \, d\lambda}.$$

The preceding representations not only make it clear that $y \in C^1(a, b)$, but in addition it follows that

$$|v^{(n_j)}(t_j)|^2 \cdot y'(t_j) = G^{(n_j+1)}(t_j, s; x)\, v^{(n_j)}(t_j) - G^{(n_j)}(t_j, s; x)\, v^{(n_j+1)}(t_j),$$

$$y'(t_i) = G^{(n_i)}(t_i, s; x).$$

Suppose now that $y'(t^*) = 0$ for some $t^* \in (a, b)$. Then $t^* \neq t_i$, because of the differential properties of G. Set $\alpha = G(t^*, s; x)/v(t^*)$ if $t^* \notin \operatorname{comp}(x)$, and set $\alpha = G^{(n_j)}(t_j, s; x)/v^{(n_j)}(t_j)$ if $t^* = t_j$. From the various representations of $y(t)$ it follows that

$$z(t) \equiv G(t, s; x) - \alpha v(t)$$

either has a double zero at $t^* \in [a, b]\backslash\operatorname{comp}(x)$ or else $z(t)$ has $n_j + 2$ zeros at some $t_j \in (a, b)$. In particular, $z(t)$ has $k + 1$ zeros in $[a, b]$. Applying the argument in Bates and Gustafson [1, p. 334] it follows that $z(t) \equiv 0$. Therefore, $y(t) = \alpha$, $a \leqslant t \leqslant b$. Letting $t = t_i$, we obtain $\alpha = 0$, which is a contradiction to $G^{(n_j)}(t_j, s; x) \neq 0$.

This shows that $y'(t) \neq 0$ on $a < t < b$, therefore $\max\{|y(t)|: a \leqslant t \leqslant b\} = \max\{|y(a)|, |y(b)|\}$. This proves inequalities (*a*) and (*b*) for $s \in [a, b]\backslash\operatorname{comp}(x)$; the validity for $s \in \operatorname{comp}(x)$ follows by continuity. ∎

Remark. Let $u_{i,j}(t)$, $0 \leqslant i \leqslant n_i - 1$, be the unique basis of $Ku = 0$ which satisfies $\mathscr{L}_{x^*}(u_{ij}) = e_p$, $p = \sum_{r<i} n_r + j + 1$.

Assume that $\mathscr{F} \geqslant 0$ is continuous on $[a, b] = \operatorname{co}\{\operatorname{comp}(x)\}$ and ψ is a particular solution of $K\psi = \mathscr{F}$. Then

$$y(t) = \psi(t) - \sum_{i=0}^{\nu} \sum_{j=0}^{n_i-1} \psi^{(j)}(t_i)\, u_{i,j}(t)$$

is the unique solution of $Ky = \mathscr{F}$, $\mathscr{L}_{x^*}y = 0$. Furthermore, $G^{(n_r)}(t_r, s; x)$ is one-signed on $a < s < b$, therefore

$$\int_a^b |G^{(n_r)}(t_r, s; x)|\, \mathscr{F}(s)\, ds$$

$$= |y^{(n_r)}(t_r)| = \left| \psi^{(n_r)}(t_r) - \sum_{i=0}^{\nu} \sum_{j=0}^{n_i-1} \psi^{(j)}(t_i)\, u_{i,j}^{(n_r)}(t_r) \right|. \tag{6.1}$$

This method of evaluation of the integral on the left in (6.1) is practical provided most of the numbers $\psi^{(j)}(s_i)$ are zero. Regardless, it is a useful numerical formula, because it replaces the indicated integration by computation of solution values at specific points.

Let F be a finite subset of $\mathbb{R}^k$ such that $[a, b] = \operatorname{co}\{\operatorname{comp}(x)\}$ for all $x \in F$. Assume $\mathscr{F} \subseteq \mathbb{R}^k$, $[a, b] = \operatorname{co}\{\operatorname{comp}(x)\}$ for all $x \in \mathscr{F}$, and $F = \bar{\mathscr{F}} - \operatorname{int}\{\mathscr{F}\}$.

LEMMA 6.4. *The following maximization identities are valid*:

(a) $\sup\{|G(t, s; x)|: x \in \mathscr{F}\} = \max\{|G(t, s; x)|: x \in F\}$,

(b) $\sup\{\int_a^b |G(t, s; x)|\, ds: x \in \mathscr{F}\} = \max\{\int_a^b |G(t, s; x)|\, ds: x \in F\}$ (*see Bates and Gustafson* [2]).

Of particular interest is the case when F corresponds to all two-point boundary conditions at a, b; this set is denoted by F_2. Similarily, F_k denotes the set corresponding to all k-point boundary conditions.

LEMMA 6.5. *Let K be disconjugate on $[c, d]$, $c \leqslant a < b \leqslant d$. Assume $x = x(\alpha, T), y = y(\alpha, T^*), t_0^* = c, t_0 = a, t_\nu = b, t_\nu^* = d, t_i = t_i^*$ for $0 < i < \nu$. then*

$$|G(t, s; x)| \leqslant |G(t, s; y)|, (t, s) \in [a, b] \times [a, b].$$

The inequality is strict when $|a - c| + |b - d| > 0$ and

$$(t, s) \in \{[a, b]\backslash\text{comp}(x)\} \times (a, b).$$

Proof. Assume $|a - c| > 0$, $|b - d| = 0$. For fixed $(t, s) \in [a, b] \times [a, b] \times [a, b]$ the function $t_0 \to G(t, s; x)$ is differentiable and

$$\frac{\partial G}{\partial t_0}(t, s; x) = U^*(t)\left\{\frac{\partial Z}{\partial t_0} Z^{-1}\mathscr{L}_{x^*}[\mathscr{K}(\cdot, s)] - M[\mathscr{K}(\cdot, s)]\right\},$$

$$Mu = ((u'(t_0),\ldots, u^{(\alpha_0)}(t_0), 0,\ldots, 0)^T,$$

$$Z = \mathscr{L}_{x^*}(U), x^* = (t_0,\ldots, t_0,\ldots, t_\nu,\ldots, t_\nu).$$

Define $u(t) = (\partial G/\partial t_0)(t, s; x)$, s and x fixed, then $u(t)$ is a solution of $Ku = 0$, $\mathscr{L}_{x^*}u = (0,\ldots, 0, -G^{(\alpha_0)}(t_0, s), 0,\ldots, 0)^T$. Since sign $\{G^{(\alpha_0)}(t_0, s)\} = (-1)^{k-\alpha}$ on $t_0 < s < t_\nu$ it follows that

$$\prod_{i=0}^{\nu} (t - t_i)^{\alpha_i} u(t) < 0, \qquad t \in \text{comp}(x).$$

Given $t \in [a, b]\backslash\text{comp}(x)$, $a < s < b$, set $\beta = \text{sign}\,\{\prod_{i=0}^{\nu}(t - t_i)^{\alpha_i}\}$. By the mean value theorem,

$$\beta[g(t, s; x) - G(t, s; y)] = \beta\frac{\partial G}{\partial t_0}(a - c) < 0$$

for some $t_0 \in (c, a)$. However, $\beta G(t, s; x) \geqslant 0$, therefore

$$|G(t, s; x)| < |G(t, s; y)|.$$

A similar calculation can be completed when $|a - c| = 0$, $|b - d| > 0$, and we find that $Ku = 0$, $\mathscr{L}_{x^*}u = (0, \ldots, 0, -G^{(\alpha_\nu)}(t_\nu, s))^T$. In this case,

$$\prod_{i=0}^{\nu} (t - t_i)^{\alpha_i} u(t) > 0, \qquad t \notin \text{comp}(x),$$

and the conclusion follows in the same way.

If $|a - c| > 0$ and $|b - d| > 0$, then two applications of the preceding results gives the desired inequality. ∎

7. Identities and Estimates for the Green's Function Associated with $(d/dt)^k$

The purpose of this section is to develop some identities and inequalities for the Green's function $\mathscr{G}(t, s; x)$ associated with the BVP $(d/dt)^k y = \mathscr{F}(t)$, $\mathscr{L}_x(y) = 0$, $x \in \mathbb{R}^k$. As usual, $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $t_0, \ldots, t_\nu$ are the distinct elements of comp(x) in increasing order, $n_i = n_x(t_i)$, $0 \leqslant i \leqslant \nu$, $a = t_0$, $b = t_\nu$.

The idea is to find an estimate for

$$\int_a^b |\mathscr{G}(t, s; x)| \, \mathscr{F}(s) \, ds \tag{7.1}$$

when $\mathscr{F}(t) \geqslant 0$. It turns out that (7.1) can be evaluated explicitly when $\mathscr{F}(t)$ is real analytic on $[a, b]$. However, this formula has a very complicated t-dependence. For large values of k it is more practical to use the general estimates of Section 6; hence there is interest in evaluation of the integrals

$$\int_a^b |\mathscr{G}^{(n_0)}(t_0, s; x)| \, \mathscr{F}(s) \, ds, \int_a^b |\mathscr{G}^{(n_\nu)}(t_\nu, s; x)| \, \mathscr{F}(s) \, ds. \tag{7.2}$$

Given in this section are explicit formulas for (7.2) in the case when $\mathscr{F}(t)$ is real analytic on $[a, b]$.

Define $U_0, \ldots, U_n$, $V_0, \ldots, V_n$ by the relation

$$\prod_{i=1}^{k} (t - x_i) = \sum_{i=0}^{k} U_i (t - t_0)^i = \sum_{j=0}^{k} V_j (t - t_\nu)^j.$$

Let $\psi(x)$ be defined by the identity

$$\psi(t) = \sum_{i=0}^{N} \psi_i (t - t_0)^i, \tag{7.3}$$

with $\psi_0, \ldots, \psi_N$ being given by the recursion relations

$$\begin{aligned} 1 &= \binom{k + N}{k} \psi_N U_k, \\ 0 &= \sum_{i=p}^{(k+p)\wedge N} \Psi_i U_{k+p-i}, \qquad 0 \leqslant p \leqslant N - 1. \end{aligned} \tag{7.4}$$

Define

$$\phi(t) = \frac{1}{k!} \prod_{i=1}^{k} (t - x_i). \tag{7.5}$$

Define $C_N(x) = \binom{k+n}{k}\psi_0$, where ψ_0 is the solution of the above recursion relation, $N = 0, 1, 2,\ldots$. The dependence of $C_N(x)$ on k is carried by the symbol x. Define $D_N(x)$ similarly, by replacing U_i by V_i in the recursions (and t_0 by t_ν).

LEMMA 7.1.

$$\sum_{r=i-p}^{k\wedge i} \binom{i}{r}\binom{k+p-i}{k-r} = \binom{k+p}{k}, \qquad k \geqslant i-p \geqslant 0, \quad p \geqslant 0.$$

Proof. Use the binomial theorem three times in the identity

$$(1+x)^{k-q}(1+x)^{p+q} = (1+x)^{k+p},$$

where $q = i - p$. Then compare the coefficients of x^k on both sides. ∎

LEMMA 7.2. *The unique solution of* $(d/dt)^k y = (t - x_1)^N$, $\mathscr{L}_x y = 0$, *is*

$$y(t) = \psi(t)\,\phi(t),$$

where ψ, ϕ *are given by* (7.3)–(7.5).

Furthermore,

$$y^{(n_0)}(t_0) = \psi_0 \frac{n_0!}{k!} \prod_{i=1}^{\nu} (t_0 - t_i)^{n_i}. \tag{7.6}$$

Proof. Since $\mathscr{L}_x\phi = 0$, it follows that $\mathscr{L}_x y = 0$. It remains to prove that $y^{(k)} = (t - t_0)^N$. By Leibnitz' rule and Lemma 7.1,

$$\begin{aligned}
y^{(k)} &= \sum_{r=0}^{k} \binom{k}{r} \psi^{(r)}(t)\,\phi^{(n-r)}(t) \\
&= \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} \left\{ \sum_{i=r}^{N} \frac{i!}{(i-r)!} \psi_i (t-t_0)^{i-r} \right\} \\
&\quad \times \left\{ \sum_{j=k-r}^{k} \frac{j!}{(j-k+r)!} U_j (t-t_0)^{j-k+r} \right\} \\
&= \sum_{p=0}^{N} \sum_{i=p}^{(k+p)\wedge N} \sum_{r=i-p}^{k\wedge i} \binom{i}{r}\binom{k+p-i}{k-r} \psi_i U_{k+p-i} (t-t_0)^p \\
&= \sum_{p=0}^{N} \binom{k+p}{k} \left[\sum_{i=p}^{(k+p)\wedge N} \psi_i U_{k+p-i} \right] (t-t_0)^p.
\end{aligned}$$

Due to the recursion relations (7.4), this equals $(t - t_0)^N$. ∎

LEMMA 7.3. *The following identities are valid*:

(a) $C_0(x) = D_0(x) = 1$;

(b) $C_N(x) = \sum \prod_{i=1}^{k} (x_i - t_0)^{p_i}, \qquad D_N(x) = \sum \prod_{i=1}^{k} (x_i - t_\nu),$

for $N \geqslant 1$, *the sums being taken over the set*

$$S_N = \{(p_1, \ldots, p_k): p_i \geqslant 0, p_1 + \cdots + p_k = N\}.$$

These sums are the homogeneous product sums h_N *formed from* $y_i = x_i - t_0$ (*or* $x_i - t_\nu$), $1 \leqslant i \leqslant k$.

Proof. It suffices to verify (a) and (b) for $C_N(x)$; the proof for $D_N(x)$ is similar.

The *homogeneous product sums* h_N and the functions

$$U_j = (-1)^j \sum_{i_1 < \cdots < i_j} y_{i_1} \cdots y_{i_j}$$

are defined by the following:

$$\prod_{j=1}^{k} (1 - ty_j) = \sum_{i=0}^{k} U_i t^{k-i}, \tag{7.7}$$

$$\left[\prod_{j=1}^{k} (1 - ty_j)\right]^{-1} = \sum_{r=0}^{\infty} h_r t^r; \tag{7.8}$$

for details, see David *et al.* [6, pp. 2–3]. Of course $y_j = x_j - t_0$.

Multiply (7.7), (7.8) and collect on powers of t:

$$1 = \sum_{q=0}^{\infty} \left[\sum_{r=0 \vee (q-k)}^{q} h_r U_{k+r-q} \right] t^q.$$

Equating like powers gives

$$\begin{aligned} 1 &= h_0 U_k, \\ 0 &= \sum_{r=0 \vee (q-k)}^{q} h_r U_{k+r-q}, \qquad q \geqslant 1. \end{aligned} \tag{7.9}$$

Make the substitution $i = q - r + p$ in (7.9), then the second sum in (7.9) becomes

$$0 = \sum_{i=p}^{(k+p) \wedge (q+p)} h_{q+p-i} U_{k+p-i}.$$

If $0 \leqslant p \leqslant N-1$, then select $q = N - p$ to get

$$0 = \sum_{i=p}^{(k+p)\wedge N} h_{N-i} U_{k+p-i} \, .$$

From this relation it is clear that $\psi_i = h_{N-i}\Psi_N$ in recursions (7.4), hence the result

$$\Psi_N C_N(x) = \psi_0 = h_N \psi_N \, .$$

To prove the identity for h_N, write

$$F_N(y_1, \ldots, y_k) = \sum_{S_N} \prod_{j=1}^{k} y_j^{p_j}$$

It is easily verified that F_N and h_N are polynomials in $U_0, \ldots, U_k$. Using induction, Newton's formula (in the form $\sum_{i=1}^{k} y_i(\partial/\partial y_i) F_{N-1}(y_1, \ldots, y_k) = F_{N-1}(y_1, \ldots, y_k)$) and the recursions for h_N it can be verified that

$$\sum_{i=1}^{k} (\partial/\partial y_i)[h_N - F_N] = 0, \qquad N \geqslant 1,$$

whereby the proposition on page 455 of reference [18] shows that $h_N - F_N =$ a constant, i.e., $h_N = F_N$. ▮

Below we use the usual notation $[a, b] = \mathrm{co}\{\mathrm{comp}(x)\}$,

$$\mathrm{comp}(x) = \{t_0 < \cdots < t_\nu\}, \quad n_j = n_x(t_j), \qquad 0 \leqslant j \leqslant \nu.$$

Lemma 7.4. *Define* $\lambda_i = (t_i - a)/(b - a)$, $\bar\lambda_i = (b - t_i)/(b - a) = 1 - \lambda_i$, $0 \leqslant i \leqslant v$, *and set* $\Lambda_x = (\lambda_0, \ldots, \lambda_0, \ldots, \lambda_\nu, \ldots, \lambda_\nu)$,

$$\bar\Lambda_x = (\bar\lambda_0, \ldots, \bar\lambda_0, \ldots, \bar\lambda_\nu, \ldots, \bar\lambda_\nu), \quad n_{\Lambda_x}(\lambda_i) = n_x(t_i) = n_{\bar\Lambda_x}(\bar\lambda_i), \qquad 0 \leqslant i \leqslant \nu.$$

Then:

$$C_N(x) = (b-a)^N C_N(\Lambda_x), \qquad D_N(x) = (a-b)^N D_N(\bar\Lambda_x). \tag{7.10}$$

Proof. The functions $C_N(x)$ and $D_N(x)$ are homogeneous of degree N. ▮

Lemma 7.5. *Let* $\Lambda, \bar\Lambda$ *be defined as in the previous lemma. Define* $\lambda = (t - a)/(b - a)$. *Then:*

$$\int_a^b \mathscr{G}(t, s; x) \frac{(s-a)^N}{N!} \, ds = \frac{(b-a)^{N+k}}{(N+k)!} \cdot \left[\sum_{i=0}^{N} C_{N-i}(\Lambda_x)\lambda^i\right] \prod_{j=0}^{v} (\lambda - \lambda_j)^{n_j}, \tag{7.11}$$

$$\int_a^b \mathscr{G}(t, s; x) \frac{(b-s)^N}{N!} \, ds = \frac{(b-a)^{N+k}}{(N+k)!} \left[\sum_{i=0}^{N} D_{N-i}(\bar\Lambda_x)(\lambda - 1)^i\right] \prod_{j=0}^{\nu} (\lambda - \lambda_j)^{n_j}. \tag{7.12}$$

Proof. We illustrate with (7.11); (7.12) is similar. The LHS of (7.11) equals $y(t) = \psi(t)\phi(t)$, by Lemma 7.2. Further, $C_{N-i}(x) = \psi_i/\psi_N$, $\psi_N = \binom{N+k}{k}$, $\psi(t) = \sum_{j=0} \psi_j(t-a)^j$, hence the RHS of (7.11) by virtue of Lemma 7.4. ▮

LEMMA 7.6. *The following relations are valid for* $0 \leqslant N < \infty$:

$$\int_a^b | \mathscr{G}^{(n_0)}(t_0, s; x)| \frac{(s-t_0)^N}{N!} ds = \frac{n_0!}{(k+N)!} \left| C_N(x) \prod_{i=1}^{\nu} (t_0 - t_i)^{n_i} \right|; \tag{7.13}$$

$$\int_a^b | \mathscr{G}^{(n_\nu)}(t_\nu, s; x)| \frac{(t_\nu - s)^N}{N!} ds = \frac{n_\nu!}{(k+N)!} \left| D_N(x) \prod_{i=0}^{\nu-1} (t_\nu - t_i)^{n_i} \right|. \tag{7.14}$$

Proof. Only (7.13) is proved; (7.14) is similar. First, $\mathscr{G}^{(n_0)}(t_0, s; x)$ is of constant sign, therefore the LHS of (7.13) is just $| y^{(n_0)}(t_0)|$, using the notation of Lemma 7.2. Since $\psi_0 = \psi_N C_N(x)$, identity (7.6) completes the proof. ▮

LEMMA 7.7. *The functions* v_0, v_ν *for* $\mathscr{G}(t, s; x)$ *are given by*

$$v_0(t)/v_0^{(n_\nu)}(t_\nu) = \frac{(t-t_\nu)^{n_\nu}}{n_\nu!} \cdot \left(\frac{t-t_0}{t_\nu - t_0}\right)^{n_0-1} \cdot \prod_{j=1}^{\nu-1} \left(\frac{t-t_j}{t_\nu - t_j}\right)^{n_j},$$

$$v_\nu(t)/v_\nu^{(n_0)}(t_0) = \frac{(t-t_0)^{n_0}}{n_0!} \cdot \left(\frac{t-t_\nu}{t_0 - t_\nu}\right)^{n_0-1} \cdot \prod_{j=1}^{\nu-1} \left(\frac{t-t_j}{t_0 - t_j}\right)^{n_j}.$$

THEOREM 7.8. *Let* $\mathscr{F} \geqslant 0$ *be a real power series* $\sum_{N=0}^{\infty} \mathscr{F}_N[(t-a)^N/N!]$ *uniformly convergent in* $[a, b]$. *Then*:

$$\int_a^b | \mathscr{G}(t, s; x)| \mathscr{F}(s)\, ds = \left| \sum_{N=0}^{\infty} \frac{(b-a)^{N+k}}{(N+k)!} \mathscr{F}_N \left\{ \sum_{i=0}^{N} C_{N-i}(\Lambda_x)\lambda^i \right\} \right| \prod_{j=0}^{\nu} | \lambda - \lambda_j |^{n_j}, \tag{7.15}$$

$$\int_a^b | \mathscr{G}^{(n_0)}(a, s; x)| \mathscr{F}(s)\, ds = n_0! \left| \sum_{N=0}^{\infty} \frac{(b-a)^{N+k}}{(N+k)!} \mathscr{F}_N C_N(\Lambda_x) \right| \prod_{i=1}^{\nu} | t_i - t_j |^{n_j}. \tag{7.16}$$

(*The symbols* λ, λ_j, Λ_x, C_{N-i} *etc., are defined above.*) *Similar relations hold when* $(t-a)$ *is replaced by* $(b-t)$, n_0 *by* n_ν.

Proof. Since $\mathscr{G}(t, s; x)$ and $\mathscr{G}^{(n_0)}(a, s; x)$ are one-signed on $a < s < b$, it suffices to prove (7.15), (7.16) with the absolute value symbols removed. The formulas are therefore obtained by multiplying (7.11) and (7.13) by $\mathscr{F}_N$ and summing over N. ∎

LEMMA 7.9. *Assume* $x = (a,..., a, b,..., b) \in \mathbb{R}^k$, $n_x(a) = l$, $a < b$. *Then*

$$C_N(x) = \binom{k-l-1+N}{k-l-1}(b-a)^N, \qquad D_N(x) = \binom{l-1+N}{l-1}(a-b)^N.$$

Proof. Expand $[1-\lambda]^{-m}$ in a Maclurin series, set $m = k - l$, $\lambda = (b-a)t$ and compare with the generating function identity (7.8) for $\{h_i\}_{i=1}^{\infty}$. Since $h_N = C_N(x)$, this establishes the first formula; the second is similar. ∎

LEMMA 7.10. *Assume* $x = (a,..., a, c,..., c, b,..., b) \in \mathbb{R}^k$, $n_x(a) = p$, $n_x(c) = q$, $n_x(b) = r$, $a < c < b$. *Then*

$$C_N(x) = \left\{\sum_{s=0}^{N}\binom{q-1+s}{q-1}\binom{r-1+N-s}{r-1}\left[\frac{c-a}{b-a}\right]^s\right\}(a-b)^N,$$

$$D_N(x) = \left\{\sum_{s=0}^{N}\binom{q-1+s}{q-1}\binom{p-1+N-s}{p-1}\left[\frac{b-c}{b-a}\right]^s\right\}(b-a)^N.$$

Proof. Proceed as in the previous lemma, using Taylor's theorem and Leibnitz' rule on $[1-(c-a)t]^{-q}[1-(b-a)t]^{-r}$ and

$$[1-(a-b)t]^{-p}[1-(c-b)t]^{-q}. \quad ∎$$

LEMMA 7.11. *Let $\mathscr{G}$ be the Green's function for $(d/dt)^2$ on $[a, b]$, then*

$$\int_a^b \mathscr{G}(t, s)\frac{(s-a)^N}{N!}\,ds = \frac{(b-a)^{N+2}}{(N+2)!}\cdot\lambda\cdot[\lambda^{N+1}-1],$$

$$\int_a^b \mathscr{G}(t, s)\frac{(b-s)^N}{N!}\,ds = \frac{(b-a)^{N+2}}{(N+2)!}\cdot(1-\lambda)\cdot[(1-\lambda)^{N+1}-1],$$

where $\lambda = (t-a)/(b-a)$.

Remark. In a specific application, one can compute $U_0,..., U_k$ from (7.7) and then refer to the tables in [6] to find the homogeneous product sum h_N. For example,

$$h_1 = U_{k-1}$$

$$h_2 = U_{k-1}^2 - U_{k-2},$$

$$h_3 = U_{k-1}^2 - 2U_{k-1}U_{r-2} + U_{k-3}.$$

Other formulas for h_N are recorded in [6]. It is often convenient to use the relations

$$h_1 = \sum_{j=1}^{k} y_j ,$$

$$h_2 = \sum_{j=1}^{k} \sum_{i=j}^{k} y_i y_j ,$$

$$h_3 = \sum_{i_1 \leqslant i_2 \leqslant i_3} y_{i_1} y_{i_2} y_{i_3} ,\ldots .$$

Equally useful is the recursion relation (7.8), used in conjunction with Taylor series expansions. Finally, h_N is expressible as the $N \times N$ determinant formed from the infinite shift matrix

$$\begin{bmatrix} U_k & U_{k-1} & U_{k-2} & \cdots & U_0 & 0 & 0 & \cdots \\ 0 & U_k & U_{k-1} & \cdots & U_1 & U_0 & 0 & \cdots \\ 0 & 0 & U_k & \cdots & U_2 & U_1 & U_0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix}$$

by deleting all but columns 2, 3,..., $N + 1$ and rows 1, 2,..., N.

8. Uniqueness Estimates

Consider the multipoint problem $KMy - Qy = 0$, $L_x y = 0$. Obtained here are necessary conditions for nonuniqueness. The techniques complement those of Hartman [9], and Willett [20].

Let $x \in \mathbb{R}^{m+k}$, $[a, b] = \text{co}\{\text{comp}(x)\}$, and assume $s_1 < \cdots < s_{\mu-1}$ are those components of x for which $n_x(s_i) \geqslant m + 1$, $a < s_i < b$. Put $[s_0, s_\mu] = I_y$. Then $s_1, \ldots, s_{\mu-1}$ belong to the inherited interval I_y and the inherited boundary operator $\mathscr{L}_v$ contains $n_x(s_i) - m$ entries involving s_i, $1 \leqslant i \leqslant \mu - 1$.

Denote by $\mathscr{F}_{x,v}$ the (finite) set of all $z \in \mathbb{R}^k$ having the following properties:

(1) $\text{comp}(z) = \{s_0, \ldots, s_\mu\}$;

(2) $\max\{1, n_x(a) - m\} \leqslant n_z(s_0) \leqslant n_x[a, s_1)$;

(3) $\max\{1, n_x(b) - m\} \leqslant n_z(s_\mu) \leqslant n_x(s_{\mu-1}, b]$;

(4) $n_x(s_i) - m \leqslant n_z(s_i) \leqslant n_v(s_{i-1}, s_{i+1})$, $1 \leqslant i \leqslant \mu - 1$.

If no component of x is repeated $m + 1$ times, then $\mu = 1$, $[s_0, s_1] = I_y$, $\text{comp}(z) = \{s_0, s_1\}$ and $1 \leqslant n_x(s_0) \leqslant k - 1$, $1 \leqslant n_z(s_1) \leqslant k - 1$; in particular, (2), (3), (4) cannot be used for $\mu = 1$.

Let $\mathscr{E}_x\colon \mathbb{R}^k \to \mathbb{R}^k$ be defined as follows: $\mathscr{E}_x z$ is z with the endpoints of $\text{co}\{\text{comp}(z)\}$ replaced by a and b. Define

$$\mathscr{F}_x = \{\mathscr{E}_x z\colon z \in \mathscr{F}_{x,v}\}.$$

LEMMA 8.1. *$\mathscr{F}_x$ is a finite set which depends only on x, m, k.*

For example, given $x = (a, a, a, b, b, b)$, $m = 2$, $k = 4$, the set $\mathscr{F}_x = \{(a, a, a, b), (a, a, b, b), (a, b, b, b)\}$. However, given $x = (a, c, c, c, b, b)$, $m = 2$, $k = 4$, with $a < c < b$, the set $\mathscr{F}_x = \{(a, a, c, b), (a, c, c, b), (a, c, b, b)\}$.

LEMMA 8.2. *If $n_x(s) \leqslant m$ for all $s \in \text{comp}(x)$, then $\mathscr{F}_x$ corresponds to all two-point boundary conditions at a, b.*

If $n_x(a) \geqslant m + 1$, $n_x(b) \geqslant m + 1$, then $\mathscr{F}_x$ corresponds to all two-point conditions $z \in \mathbb{R}^k$ with $n_x(a) \leqslant n_z(a) + m$, $n_x(b) \leqslant n_x(b) + m$.

LEMMA 8.3. *Assume $KMy - Qy = 0$ has a solution y with $\mathscr{L}_x y = 0$ and K is disconjugate on $[a, b] = \text{co}\{\text{comp}(x)\}$. Then for $t \in I_y$:*

$$|[My](t)| \leqslant \max \left\{ \int_a^b | G(t, s; z)| \, |[Qy](s)| \, \chi_{I_y}(s) \, ds : z \in \mathscr{F}_x \right\}. \tag{8.1}$$

Proof. Let $\mathscr{L}_v$ be the operator inherited from M, $\mathscr{L}_x$, y and denote by I_y the inherited interval, $I_y = [s_0, s_\mu]$. The partial inversion lemma implies

$$|[My](t)| \leqslant \int_{s_0}^{s_\mu} | G(t, s; v)| \, |[Qy](s)| \, ds.$$

The RHS of this inequality does not exceed

$$\max \left\{ \int_{s_0}^{s_\mu} | G(t, s; z)| \, |[Qy](s)| \, ds : z \in \mathscr{F}_{x,t} \right\},$$

by virtue of Lemma 6.4. By Lemma 6.5, the latter does not exceed the RHS of (8.1). ∎

Assume that $v \in \mathbb{R}^m$, $y \in \mathbb{C}^m[a, b]$, $\text{comp}(v) \subseteq [a, b]$ and $\mathscr{L}_v y = 0$. Following the methods of Coppel [5], there exists functions $\phi_0, \ldots, \phi_m$ from $[a, b]$ into $[0, \infty)$ depending only on m, a, b such that

$$| y^{(i)}(u)| \leqslant \phi_i(u) \, \| y^{(m)} \|, \qquad u \in [c, d], \tag{8.2}$$

where $[c, d] = \text{co}\{\text{comp}(v)\}$ and $\|f\| = \max\{|f(s)| : c \leqslant s \leqslant d\}$.

The method of application of these ideas within this paper will be restricted to the case where $\text{comp}(v) \subseteq I_y \subseteq [a, b]$ and $\text{comp}(v) \subseteq \text{comp}(x)$. The extra information provided by these inclusions allows one to determine $\phi_0, \ldots, \phi_m$ in closed form, usually by elementary calculus inequalities, following the ideas of Hartman [9].

A theoretical method exists for the determination of ϕ_i. The idea is to write

$$y^{(i)}(t) = \int_0^d G^{(i)}(t, s; v) \, y^{(m)}(s) \, ds,$$

then use elementary estimates and monotonicity to find an upper bound for $\int_c^d |G^{(i)}(t, s; v)|\, ds$ valid for $a \leqslant t \leqslant b$, and all possible $v \in \mathbb{R}^m$ relevant to the problem. This upper estimate is called $\phi_i(t)$, hence (8.2). Such estimates have been obtained by Ostroumov [13] in the special case when $\mathscr{L}_y$ is a two-point condition. General estimates are unknown.

The differential expressions Qy and My admit estimates of the following type on each inherited interval:

$$|[Qy](s)| \leqslant \| y^{(m)} \| \sum_{i=0}^{m} | q_i(s)\, \phi_i(s)|; \tag{8.3}$$

$$| y^{(m)}(s)| \leqslant |[My](s)| + \| y^{(m)} \| \sum_{i=0}^{m-1} | a_i(s)\, \phi_i(s)|. \tag{8.4}$$

Unfortunately, the inherited interval is not explicitly known in a given application, in general, therefore estimation techniques must be employed to transfer all inequalities back to $[a, b]$. To illustrate what can be done, we state without proof the following result.

LEMMA 8.4. *Let $\phi_0, ..., \phi_m$ be the functions appearing in* (8.2) *and assume that $KMy - Qy = 0$, $\mathscr{L}_x y = 0$ has a solution $y \not\equiv 0$ on $[a, b] =$* co{comp(x)}.
If $\mathscr{L}_x$ is m-hereditary and K is disconjugate on $[a, b]$, then

$$1 \leqslant \sum_{i=0}^{m-1} \| a_i\phi_i \| + \sum_{j=0}^{m} \max \left\{ \left\| \int_a^b | G(\cdot, s; z) \| q_j(s) | \phi_j(s)\, ds \right\| : z \in \mathscr{F}_x \right\} \tag{8.5}$$

where $\| f \| = \max\{| f(t)| : a \leqslant t \leqslant b\}$.

A minor refinement of (8.5) is possible when M is given in factored form. This refinement rests upon a generalization of the Levin-type inequalities (8.2) of the form

$$| y^{(i)}(u)| \leqslant \Phi_i(u) \| My \|, \qquad 0 \leqslant i \leqslant m. \tag{8.6}$$

Again, these inequalities are valid for $u \in [c, d] \subseteq [a, b]$ where $\mathscr{L}_v y = 0$, $v \in \mathbb{R}^m$, $[c, d] =$ co{comp(v)} and $\| f \| = \max\{| f(t)| : t \in [c, d]\}$. The functions Φ_i are defined and continuous on $[a, b]$, $0 \leqslant i \leqslant m$, and depend only on m, i, a, b and the coefficients of the factored form of M.

COROLLARY 8.5. *Let $\Phi_0, ..., \Phi_m$ be the functions appearing in* (8.6). *Under the same hypotheses as the preceding lemma,*

$$1 \leqslant \sum_{j=0}^{m} \max \left\{ \left\| \int_a^b | G(\cdot, s; z) \| q_i(s) | \Phi_j(s)\, ds \right\| : z \in \mathscr{F}_x \right\}. \tag{8.7}$$

The incidence of usefulness of the corollary is substantially lower than that the lemma. This is due to the fact that factored forms are most often quite complicated, hence Φ_i is itself complicated.

A third refinement of (8.5) is possible. The idea is to use the Green's function for M to invert the partial inversion formula itself, obtaining a closed-form expression for y. This procedure will be successful provided there exists $v^* \in \mathbb{R}^m$, $\text{comp}(v^*) \subseteq \text{comp}(x)$, such that $\mathscr{L}_{v^*}y = 0$ and $I_y = \text{co}\{\text{comp}(v^*)\}$. Advantages of this refinement are: (1) it lowers the orders of the Green's functions appearing in the final identity; (2) estimates of the Green's function for M in terms of simpler Green's functions are sometimes possible (see the next section).

9. Uniqueness for Two-Point Problems

The purpose of this section is to show how to apply the results of preceding sections to the $(l, n - l)$-boundary value problem

$$Ly = 0, \;\; y^{(i)}(a) = 0, \;\; y^{(j)}(b) = 0, \;\; 0 \leqslant i \leqslant l - 1, \;\; 0 \leqslant j \leqslant n - l - 1, \quad (9.1)$$

where L is an nth order linear ordinary differential operator. Sought are inequalities which guarantee uniqueness of the zero solution to (9.1).

The idea in the applications is to select operators K, M, Q which decompose L as $L = KM - Q$, such that K and M are disconjugate, while Q has order $\leqslant m$. The boundary operator $\mathscr{L}_x$ is specified by $x = (a,\ldots, a, b,\ldots, b) \in \mathbb{R}^n$, $n = k + m$, $n_x(a) = l$, $n_x(b) = n - l$.

LEMMA 9.1. *The following are true for a decomposition $L = KM - Q$ of* (9.1):

(a) *The operator $\mathscr{L}_x$ is m-hereditary iff* $\max\{l - m, k - 1\} > 0$.

(b) *Every inherited interval is $[a, b]$ iff $m < l < k$.*

(c) *The set $\mathscr{F}_x$ consits of all $z = (a,\ldots, a, b,\ldots, b) \in \mathbb{R}^k$ such that*

$$\max\{1, l - m\} \leqslant n_z(a) \leqslant \min\{k - 1, l\}.$$

In any case, $\mathscr{F}_x$ contains at most $k - 1$ elements.

The above lemma is valid assuming only that M is disconjugate and $y \to (Ky, \mathscr{L}_v y)$ has kernel zero ($\mathscr{L}_v$-the inherited operator). This observation may be useful for certain special equations, especially when K fails to be disconjugate on $[a, b]$.

The following lemma is similar to Lemma 1, page 131, of Inozemtseva [11]; the proof is analogous.

LEMMA 9.2. *Let $G_l(t, s)$, $\bar{G}_l(t, s)$ be the Green's functions on $[a, b]$ for the $(l, k - l) - BVP$ associated with the disconjugate operators $(d/dt)^k + (-1)^{k-l}A(t)$,*

$(d/dt)^k + (-1)^{k-l}\bar{A}(t)$, respectively, where $A, \bar{A} \in C[a, b]$ and $A(t) \geqslant \bar{A}(t)$. Then

$$|G_l(t, s)| \leqslant |\bar{G}_l(t, s)|, \qquad a \leqslant t, \quad s \leqslant b.$$

EXAMPLE 9.3. $Ly \equiv y^{(6)} - p_2(t)y'' - p_1(t)y' - p_0(t)y$, $x = (a, a, a, a, b, b)$.

The decomposition $L = KM - Q$ is chosen so that $K = (d/dt)^4$, $M = (d/dt)^2$. An application of the first lemma shows $\mathscr{L}_x$ is m-hereditary and $\mathscr{F}_x = \{(a, a, b, b), (a, a, a, b)\}$.

The inequalities $|y^{(i)}(s)| \leqslant \phi_i(s) \| y'' \|$, $0 \leqslant i \leqslant 2$, are satisfied by $\phi_0(s) = (s - a)^2/2$, $\phi_1(s) = s - a$, $\phi_2(s) = 1$. Insertion of these estimates into the uniqueness inequality of Section 8 gives the following sufficient condition for uniqueness:

$$\frac{377.4}{10^6}(b - a)^6 \| p_0 \| + \frac{154.91}{10^6}(b - a)^5 \| p_1 \| + \frac{9}{2048}(b - a)^4 \| p_2 \| < 1. \tag{9.2}$$

Optimization of the $\| p_2 \|$-term is obtained in (9.2), compared with other criteria. On the other hand, (9.2) can be improved if $p_1 \equiv 0$ or $p_2 \equiv 0$, as is shown by the following.

EXAMPLE 9.4. $Ly \equiv y^{(6)} - p_0(t)y$, $x = (a, a, a, a, b, b)$.

Take $K = (d/dt)^6$, $M = Id$, then $\mathscr{F}_x = \{x\}$ and by Lemmas 8.4, 7.5 a sufficient condition for uniqueness is:

$$(30.4)\frac{(b - a)^6}{10^6} \| p_0 \| < 1. \tag{9.3}$$

A corresponding estimate of integral type is

$$\frac{9}{1024}(b - a) \int_a^b (s - a)(b - s)^3 \, |p_0(s)| \, ds < 1. \tag{9.4}$$

EXAMPLE 9.5. $Ly \equiv (y'' - q(t)y)^{iv} - p_0(t)y$, $x = (a, a, a, b, b, b)$.

Select $My = y'' - q(t)y$, $Ku = u^{iv}$. It is presupposed that $q(t) \geqslant 0$ on $[a, b]$, making M disconjugate. Each inherited interval is $[a, b]$ and $\mathscr{F}_x = \{(a, a, a, b), (a, a, b, b), (a, b, b, b)\}$.

Denote by $G(t, s)$ the Green's function for M and by $\mathscr{G}(t, s)$ the Green's function for $(d/dt)^2$, then by Lemma 9.2 $|G(t, s)| \leqslant |\mathscr{G}(t, s)|$. By virtue of the partial inversion formula, possible nontrivial solutions y of $Ly = 0$, $\mathscr{L}_x y = 0$ must satisfy

$$My = \int_a^b G(t, s; z) \, p_0(s) \, y(s) \, ds.$$

Since y vanishes at a and b,

$$y(t) = \int_a^b \mathscr{G}(t, s) \int_a^b G(s, r; z)\, p_0(r)\, y(r)\, dr\, ds.$$

The usual integral estimates give the following sufficient condition for uniqueness:

$$\max \left\{ \max_{a \leqslant t \leqslant b} \int_a^b |\, \mathscr{G}(t, v)| \int_a^b |\, G(v, r; z)|\, |\, p_0(r)|\, dr\, dv \colon z \in \mathscr{F}_x \right\} < 1.$$

One condition which arises in this manner is

$$\frac{(b - a)^6 \| p_0 \|}{2160} < 1. \tag{9.5}$$

Due to the arbitrary nature of $q(s)$, the seemingly more natural choice $K = (y'' - q(t)\, y)^{iv}$, $M = Id$, leads to an intractable problem in the estimation of the Green's function.

In any case, (9.5) must be viewed as a compromise due to errors made in final estimates. For example, if $q(t) \equiv 0$, then (9.3) is considerably better than (9.5).

Example 9.6. $Ly \equiv (y'' - q(t)y)'' - p_1(t)y' - p_0(t)y$, $x = (a, a, a, b)$.

Consider the adjoint problem $L^*v = v^{iv} - q(t)v'' + (p_1(t)v)' - p_0(t)v = 0$, $x^* = (a, b, b, b)$. It is well-known that $Ly = 0$, $\mathscr{L}_x y = 0$ has $y = 0$ as its only solution iff $L^*v = 0$, $\mathscr{L}_{x_0} v = 0$ has $v \equiv 0$ as its only solution. Assume $q(t) \geqslant 0$, $q \in C^2[a, b]$.

Select $M = (d/dt)^2$, $K = (d/dt)^2 - q(t)$, then $\mathscr{F}_x$ is the singleton (a, b) and each inherited interval contains b, but not a. Possible solutions of $L^*v = 0$, $\mathscr{L}_{x_0} v = 0$ satisfy

$$|\, v''(t)| \leqslant \int_a^b |\, \mathscr{G}(t, r)|\, |\, p_1'(r)\, v(r) + p_1(r)\, v'(r) - p_0(s)\, v(r)|\, \chi_{I_v}(r)\, dr, \tag{9.6}$$

by virtue of the uniqueness estimates and Lemma 9.2, where $\mathscr{G}$ is the Green's function for $(d/dt)^2$ on $[a, b]$.

Relation (9.6) gives the following sufficient conditions for uniqueness, assuming $q \in C^2$, $p_1 \in C^1$, $p_0 \in C$:

$$\int_a^b |\, p_1'(r) - p_0(r)|\, \frac{(b - r)^3}{2}\, dr + \int_a^b |\, p_1(r)|\, (b - r)^2\, dr < 1;$$

$$\frac{(b - a)^4}{3 2^3 4^{1/2}} \| p_1' - p_0 \| + \frac{(b - a)^3}{9 2^3 1^{/2}} \| p_1 \| < 1.$$

10. Uniqueness for Three-Point Problems

The purpose of this section is to indicate how to apply the partial inversion methods to the study of uniqueness for the *BVP*

$$\begin{gathered} Ly = 0, \qquad y^{(i)}(a) = y^{(j)}(c) = y^{(k)}(b) = 0, \qquad 0 \leqslant i \leqslant p-1, \\ 0 \leqslant j \leqslant q-1, \qquad 0 \leqslant k \leqslant n-p-q-1, \end{gathered} \tag{10.1}$$

where L is an nth order linear ordinary differential operator and $a < c < b$.

Following the development for the two-point *BVP*, we seek to decompose L as $KM - Q$ where K and M are disconjugate, while Q has order $\leqslant m$. The boundary operator $\mathscr{L}_x$ is specified by $x = (a,..., a, c,..., c, b,..., b) \in \mathbb{R}^n$, with $n_x(a) = p$, $n_x(c) = q$, $n_x(b) = n - p - q$, $n = m + k$.

Lemma 10.1. *Decompositions* $L = KM - Q$ *of* (10.1) *satisfy*:

(a) *The operator* $\mathscr{L}_x$ *is m-hereditary iff either* $k > p$ *and* $q = m$, *or else* $\max\{p - m, q - m, k - p - q\} > 0$.

(b) *Every inherited interval is* $[a, b]$ *iff* $m < p < k - q$.

(c) *If* $q \leqslant m$, *then* $\mathscr{F}_x$ *consists of all* $z = (a,..., a, b,..., b) \in \mathbb{R}^k$ *with* $\max\{1, p - m\} \leqslant n_z(a) \leqslant \min\{p + q, k - 1\}$.

(d) *If* $q > m$, *then* $\mathscr{F}_x$ *consists of all* $z = (a,..., a, c,..., c, b,..., b) \in \mathbb{R}^k$ *with* $\max\{1, p - m\} \leqslant n_z(a) \leqslant p$, $q - m \leqslant n_z(c) \leqslant \min\{m, p\} + \min\{k - p, q\}$, $p + q - m \leqslant n_z(a) + n_z(c) \leqslant \min\{k - 1, p + q\}$.

Example 10.2. $x = (a, a, a, c, c, c, b)$, $a < c < b$,

$$Ly \equiv y^{vii} - p_0(t)y - p_1(t)y' - p_2(t)y''.$$

Choose $m = 2$, $k = 5$, then $\mathscr{F}_x$ consists of the points (a, a, a, c, b), (a, a, c, c, b), (a, c, c, c, b). Select $M = (d/dt)^2$, $K = (d/dt)^5$. The division ratios are $\lambda_0 = 0$, $\lambda_1 = (c - a)/(b - a)$, $\lambda_2 = 1$. Every inherited interval contains a and c, but not b.

On I_y the estimates $|y^{(2-i)}(t)| \leqslant ((t - a)^i/i!) \| y'' \|$ $(0 \leqslant i \leqslant 2)$ are valid. Some improvement of this estimate is possible because $|y^{(2-i)}(t)| \leqslant (|t - c|^i/i!) \| y'' \|$ $(0 \leqslant i \leqslant 2)$ as well. By the uniqueness estimates, possible solutions of $Ly = 0$, $\mathscr{L}_x y = 0$, must satisfy for each $t \in I_y$

$$\begin{aligned} |y''(t)|/\|y''\|_{I_y} \leqslant & \int_a^b |\mathscr{G}(t, s; z_1)| \frac{(s-a)^2}{2} ds \cdot \|p_0\| \\ & + \int_a^b |\mathscr{G}(t, s; z_2)| (s - a)\, ds \cdot \|p_1\| \\ & + \int_a^b |\mathscr{G}(t, s; z_3)|\, ds \, \|p_2\| \end{aligned}$$

where $z_1, z_2, z_3 \in \mathscr{F}_x$ and $\mathscr{G}$ is the Green's function for $(d/dt)^5$. The RHS of this inequality can be maximized over $t \in [a, b]$ and $z_1, z_2, z_3 \in \mathscr{F}_x$, using the estimates of Sections 7, 8. This procedure leads to the following sufficient condition for uniqueness:

$$\frac{(b-a)^7}{7!} \cdot \frac{27(1+3\lambda_1+6\lambda_1^2)\lambda_1^4}{64} \| p_0 \| + \frac{(b-a)^6}{6!} \cdot \frac{27(1+3\lambda_1)\lambda_1^4}{64} \cdot \| p_1 \| + \frac{(b-a)^5}{5!} \cdot \frac{27\lambda_1^4}{64} \| p_2 \| < 1.$$

This inequality can be improved by exerting more effort in the maximization process. Similarly, an absence of one or more terms could lead to dramatic changes in the constants. A third improvement is realized by using estimates $| y^{(2-i)}(t)| \leqslant \phi_i(t) \| y'' \|$, where $\phi_i(t)$ is a polynomial in $(t-a)$ satisfying

$$\min \left\{ \frac{(t-a)^i}{i!}, \frac{| t-c |^i}{i!} \right\} \leqslant \phi_i(t), \qquad 0 \leqslant i \leqslant 2.$$

For example,

$$\phi_0(t) = 1,$$

$$\phi_1(t) = \frac{2-3\lambda_1}{2-\lambda_1}(t-a) + \frac{\lambda_1^2(b-a)}{2-\lambda_1},$$

$$\phi_2(t) = \frac{\lambda_1^2}{8}(b-a)^2 + \alpha([t-a] - 1/2[c-a])^2,$$

where $2\alpha \equiv [4 - 8\lambda_1 + 3\lambda_1^2]/(2-\lambda_1)^2$, will suffice.

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